On the Playability of Violins. Part I: Reflection Functions

J. Woodhouse
Cambridge University Engineering Department

Dedicated to the memory of Lothar Cremer (1905–1990)

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Part I: Reflection Functions

Summary

The perceived quality of a violin depends not only on its sound, but on its response to the mechanical input from the player. The standard theoretical model of the bowing process is used to examine how one instrument might differ from another in this regard. The relevant aspects of the behaviour of the instrument body, at least to a first approximation, are most conveniently described in terms of two reflection functions for transverse waves on the sections of string to either side of the bowed point. The nature of these functions is studied, and the scope and limitations of models which assume the reflection functions to be narrow compared with the period of the note in question are discussed in detail. In a companion paper [1] this theoretical framework is applied to study some particular issues of "playability".

Über die Spielbarkeit von Violinen.
Teil I: Reflexionsfunktionen

Zusammenfassung

Die subjektive Qualität von Violinen hängt nicht allein von ihrem Klang ab, sondern auch von ihrer Reaktion auf die mechanische Einwirkung durch den Spieler. Mit Hilfe des theoretischen Standardmodells des Streichvorgangs wird untersucht, in welcher Weise ein Instrument sich vom anderen in dieser Hinsicht unterscheiden könnte. Die maßgeblichen Aspekte des Verhaltens des Instrumentencorpus werden, zumindest in erster Näherung, am bequemsten durch zwei Reflexionsfunktionen für Transversalwellen auf den Saitenabschnitten beiderseits des Anstreichpunktes beschrieben. Die Natur dieser Funktion wird untersucht, und die Tragfähigkeit sowie die Grenzen von Modellen, welche die Reflexionsfunktionen als kurz im Vergleich zur Periode des betreffenden Tones annehmen, werden eingehend diskutiert. In einer Parallelarbeit [1] wird dieser theoretische Rahmen auf die Untersuchung einiger Aspekte der „Spielbarkeit" angewandt.

L'aisance de jeu des violons.
Partie I: fonctions de réflexion

Sommaire

La qualité d'un violon telle qu'on la perçoit dépend non seulement des sons qu'il émet, mais aussi de sa réponse aux stimulations mécaniques de l'exécutant. A l'aide du modèle théorique standard du processus de frottement de l'archet, on examine les différences éventuelles que l'on peut trouver de ce point de vue entre les instruments. En première approximation, il semble que les aspects pertinents du comportement du corps de l'instrument puissent être commodément décrits par deux fonctions de réflexion des ondes transversales sur les portions de cordes situées de part et d'autre du point de frottement. On étudie la nature de ces fonctions, et l'on procède à une discussion détaillée de la portée et des limites des modèles qui font l'hypothèse que les fonctions de réflexion sont bornées par rapport à la période de la note jouée. Dans un article complémentaire [1] on exploite ce cadre théorique dans l'étude de quelques traits particuliers de la facilité de jeu, ou «jouabilité», des violons.

1. Introduction

There have been many efforts over the years to find physical parameters which correlate well with subjective quality judgements of violins. Most of these have concentrated on judgements of the sound of instruments by listeners, and most investigators would agree that the task proves very hard, and that the level of success to date is rather limited. Most listeners, when blindfolded, prove to be rather undiscriminating when trying to tell one instrument from another (although there is no doubt that some listeners are quite acute at this task [2]). However, the *player* of the test instruments usually has clear views on their relative merits, and there seems to be a good degree of consistency of comments from one player to another. This is hardly surprising. The non-playing listener only has access to the final result of whatever actions the player might take to compensate for the idiosyncracies of individual instruments. The player, on the other hand, is inside the feedback loop of those compensations, and so has a lot of extra information available to make comparative judgements between instruments. A sufficiently good player may be able to make a reasonable sound on almost any instrument, but will still be very

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Dr. J. Woodhouse, Cambridge University Engineering Department, Trumpington Street, Cambridge CB2 1PZ, U.K.

clear that some instruments are preferable to others because they make the job easier.

In other words, instruments differ from one another not only in sound, but also in what we may term "playability". This quality may be just as elusive as "sound" to pin down in terms of definite, physical quantities which one might be able to relate to constructional features of an instrument, but exploring it at least offers a different view on a long-standing problem. If some candidate quantities can be suggested for physical correlates of aspects of playability, there is a good chance that psychological experiments designed to test them might obtain good data by employing player judgements.

So what might be meant by "playability"? An important class of possible phenomena is best introduced by example. A very common reaction is that, rather quickly after being handed an unfamiliar instrument to try, a player may say "This one is certainly very easy to play." The judgement seems to have little to do with sound quality, and may indeed be followed by disparaging remarks about not liking the tone. The speed of producing this reaction suggests that it does not rely on very subtle judgements, associated with particular notes or specialised bowing techniques. Informal observation suggests that different players are reasonably consistent about which instruments are to be given this accolade. Presumably the player is reacting to some desirable quality in the responsiveness to bowing, and therefore is pointing to a difference between instruments in the detailed behaviour of the bow-string interaction. The main concern of this paper and its sequel [1] is to explore what the present-day understanding of the physics of the bowed string has to say about such differences. This leads to a relatively clear-cut scientific enquiry, since the influence of the dynamics of the violin string and body on the bowing process comes through a well-defined route, which will be examined in some detail.

In this paper, a theoretical framework will be established within which questions of playability may be addressed. In the companion paper, this framework is employed to study two specific problems related to playability. These are the bow-force limits for playing of steady notes, and the question of which particular regime of self-sustained oscillation might be established from a given starting transient.

Before starting the technical discussion, it should be noted that there is a second class of phenomena which might also be described in terms of "playability". These involve significant feedback from the radiated sound of the instrument via the player's ears (and perhaps tactile feedback from vibration of the instrument body through the player's fingers, chin and so on). This class of phenomena takes us back into the territory of "sound" judgements, with all the associated difficulty. Such effects are not to be ignored, but they are not the main subject of these two papers. Before leaving them aside, it is perhaps of interest to mention one possible example. Violinists commonly use vibrato when playing, and it is well known that this adds considerable "richness of tone" to the sound. This rich quality is in marked contrast to the vibrato sound from, for example, an electronic synthesiser. One reason is that the synthesiser varies only the frequency and perhaps amplitude during the vibrato cycle, while keeping the waveform constant. In a violin, by contrast, the very peaky frequency response of the body results in considerable fluctuation of spectral content during a vibrato cycle [3, 4, 5]. Now, it may happen that a particular note and its harmonics fall in such a way in relation to this body response spectrum that there is *not* very much spectral fluctuation. Such a note may well feel "dead" to the player. This is clearly a "playability" judgement of a sort, arising from aural feedback. The player will perhaps compensate by increasing the vibrato amplitude. It is not difficult to imagine constructing some kind of note-by-note "vibrato sensitivity index" from a frequency response curve, in which some weighted sum of the spectral slopes at the various harmonics of each note was calculated. Such an index could then be compared with player comments on "lively" or "dead" notes on that instrument.

2. Characterising body behaviour

The parameters controlled by the player to produce a given note are as follows: the bow speed, the position of the bow along the axial length of the string, the normal force between bow and string, and the width of bow hair in contact with the string (controlled by tilting the bow more or less on its side). All of these will be time dependent, but are likely to take more-or-less constant values during the steady portion of a note. The bow speed will be denoted v_b, the normal force (the "bow force") f_b, and the position of the bow on the string via the parameter β, defined such that if the string has length L, the bowed point is a distance βL from the bridge. Some interesting measurements of these three parameters during playing have been described by Askenfelt [6].

If the time histories of these player's parameters are specified, the consequent motion of the string is a nonlinear self-excited oscillation determined by the dynamics of the string, instrument body and bow, in conjunction with the frictional behaviour of the rosin which has been applied to the bow. Variations in playability among instruments presumably arise from differences in string motion when the player parameters, bow behaviour and rosin properties are all kept the same. To understand what governs any such differences, we must review theoretical models of the action of a bowed string. Some of this preliminary material has appeared elsewhere [7, 8, 9], but for the present purpose we require some extra details and a rather different emphasis.

To obtain a model which is reasonably tractable, it is useful to neglect for the moment the finite width of the ribbon of bow hair in contact with the string. Instead, we suppose that the bow acts at a single point. At that point it applies a force to the surface of the string, whose tangential, frictional, component is responsible for driving the transverse oscillation of the string in the plane of bowing. (We will ignore here any influence of short-term fluctuations in the normal component of this force, and any string motion in the plane orthogonal to the bow – see Cremer [9], section 5.6. We also neglect any interaction with longitudinal string motion, either through finite-amplitude effects in the string vibration or from coupling at the bridge.) The magnitude of the friction force depends in a nonlinear manner on the relative motion between bow and string, and perhaps on the past history of that motion. The nature of the nonlinear relation depends on the tribological behaviour of the rosin, and will be discussed briefly later.

The rest of the system, the dynamical behaviour of the string and body (and bow), is believed to be well represented by linear theory. The important result follows that all information about the playability of a given note on a given instrument, within this particular approximation, is contained in one response function, which determines the string-surface motion following an applied tangential force. The first task is obviously to understand the structure of this response, and which aspects of it might be expected to vary between instruments or between notes on a given instrument. The information within this response function can be expressed in several different forms, each having an advantage from some point of view. The most direct representation, requiring no further approximations, is the driving-point admittance at the bowed point on the string. This is the velocity response on the surface of the string to an applied sinusoidal force $\exp(\mathrm{i}\,\omega t)$, and we denote it $G(\omega)$. In the time domain, the inverse Fourier transform of this will be the impulse response $g(t)$.

The functions $G(\omega)$ and $g(t)$ contain contributions from two types of string motion: transverse and torsional. The bowed note generally corresponds closely in pitch to the fundamental transverse string resonance, so that it is natural to suppose that the transverse component of the response is dominant in determining the string motion. However, the torsional component is by no means negligible, and almost certainly plays a significant role in the detailed response of the string, and hence contributes to playability. Although the torsional resonance frequencies do not produce any very obvious influence, damping seems to be much higher for torsional motion than for transverse motion [10], so that scattering into torsional waves represents a significant energy loss to the system, which in turn influences the stability of motion and the behaviour during transients [8].

It will certainly be necessary to consider torsional motion further in due course, but for the moment it is convenient to ignore it, and examine the structure of the response $g(t)$ due to transverse motion only. As has been pointed out before [7, 8], it is both illuminating and much more efficient for the purposes of computer simulation of bowed string motion to express the impulse response in terms of two *reflection functions*. The string near the bowed point supports only one type of transverse travelling wave, approximately that of an ideal text-book stretched string, but modified by the effects of bending stiffness. This makes it rather easy to describe the sequence of events following an impulsive force applied at the bowed point. There will be an immediate response, which will approximate to a delta function of velocity [7]. A mirror-image pair of waves will then travel outwards from this point, which will subsequently reflect back and forth between the two ends of the string, independently of each other. They will gradually be modified in shape by dissipative and dispersive processes during propagation and reflection, and will eventually die away. The response $g(t)$ is made up of the initial spike approximating a delta function, followed by a regular series of passages of these two travelling waves.

This may be made explicit rather easily. We can define two functions $h_1(t)$ and $h_2(t)$ which contain all the information about a single reflection from the bridge and from the nut or player's finger respectively. If we imagine that the finite length of string between bow and finger is replaced by a semi-infinite length, then the impulse response would consist of just the initial spike followed by one reflection from the bridge. Denote this reflection $h_1(t)$, the *bridge reflection function*. The function is defined for a unit amplitude of initial delta function, so that for any real system, for which the impulsively excited point on the string eventually returns to its initial position, $h_1(t)$ must satisfy

$$\int_{-\infty}^{\infty} h_1(t)\,\mathrm{d}t = -1\,. \tag{1}$$

(Not all theoretical models satisfy this condition, as will be discussed later.) In a similar way we can define

$h_2(t)$, the *nut/finger reflection function.* These two functions contain all the information about dissipation and dispersion during propagation over their respective sections of string, as well as about the actual processes of reflection from the string terminations. No assumption is being made at this stage about any particular theoretical model for propagation on the string (beyond that of linear theory).

The rest of $g(t)$ simply consists of multiple reflections of the two waves (see McIntyre and Woodhouse [7], especially Fig. 2). The effect of each successive reflection may be found by convolution with the appropriate reflection function. If we denote the operation of convolution by an asterisk, then

$$g(t) = \frac{Y_0}{2}[\delta(t) + h_1(t) + h_2(t) + 2h_1 * h_2 + h_1 * h_2 * h_1 + h_2 * h_1 * h_2 + \ldots] \quad (2)$$

where Y_0 is the characteristic admittance of the string (equal to $1/\sqrt{Tm}$ for an ideal string of tension T and mass per unit length m). Writing $H_1(\omega)$ and $H_2(\omega)$ for the Fourier transforms of $h_1(t)$ and $h_2(t)$ and using the convolution theorem, we obtain,

$$\begin{aligned}\frac{2}{Y_0}G(\omega) &= 1 + H_1 + H_2 + 2H_1H_2 + H_1^2H_2 \\ &\quad + H_1H_2^2 + 2H_1^2H_2^2 + \ldots \\ &= (1 + H_1 + H_2 + H_1H_2) \\ &\quad \cdot (1 + H_1H_2 + H_1^2H_2^2 + \ldots) \\ &= \frac{1 + H_1 + H_2 + H_1H_2}{1 - H_1H_2},\end{aligned} \quad (3)$$

the final form being obtained by summing as a geometric progression.

The two reflection functions themselves have structure which can be made explicit. For example $h_1(t)$ contains the effects of propagation along the short length of string to the bridge and back, and also the effects of the reflection from the bridge. The propagation contribution would simply cause a time delay for an ideal string, but for a real string there will be a small amount of dispersion and a frequency-dependent propagation loss. The contribution from bridge reflection will consist of a mixture of decaying sinusoids, from the impulsive excitation of the vibration modes of the body. We are most interested in the influence of this bridge reflection, since variations in playability among instruments should be present even if identical strings are fitted to them all.

If the admittance presented to the string at the bridge notch (in the assumed plane of vibration, governed by the bowing direction) is $Y_1(\omega)$, and the behaviour of the string in the immediate vicinity of the notch may be approximated by that of an ideal string, then the reflection coefficient at the bridge is

$$R_1(\omega) = \frac{Y_1(\omega) - Y_0}{Y_1(\omega) + Y_0} \quad (4)$$

(see eq. (5.20) of Cremer [9]). Note that in general $Y_0 \gg |Y_1(\omega)|$ so that $R_1 \approx -1$, as we would expect for a string termination which is approximately fixed. If the short length of string between bridge and bow is represented by an ideal string, then

$$H_1(\omega) = \mathrm{e}^{-2\mathrm{i}\omega\beta L/c} R_1(\omega) \quad (5)$$

where $c = \sqrt{T/m}$ is the wave speed on the string, since propagation simply imposes a phase shift. For a real string, dispersion will produce a variation of phase with frequency which does not quite follow this simple linear form. There will also be a small amplitude reduction, especially at higher frequencies, representing propagation loss.

In the time domain, the ideal-string approximation gives $h_1(t)$ as the inverse transform of $R_1(\omega)$ delayed by $2\beta L/c$. Using the approximation $Y_0 \gg |Y_1(\omega)|$ gives a simple result for this inverse transform: from eq. (4)

$$R_1(\omega) \approx -1 + \frac{2Y_1(\omega)}{Y_0}$$

so that

$$h_1(t) \approx -\delta(t - 2\beta L/c) + \frac{2}{Y_0} y_1(t - 2\beta L/c) \qquad (t \geq 2\beta L/c) \quad (6)$$

where $y_1(t)$ is the inverse transform of $Y_1(\omega)$, in other words the impulse response at the bridge notch of the body (without effects of coupling to the string). This makes explicit the earlier description of the reflection function as containing a mixture of decaying sinusoids representing vibration modes of the violin body.

This completes the discussion of how the body behaviour influences the transverse string motion at the bowed point. It remains to point out how the torsional behaviour may be incorporated in this description. This is straightforward in principle, although not easy to carry through in practice since reliable measurements of torsional response are hard to make. Within the assumption of linear theory, torsional motion may simply be superposed on transverse motion. It corresponds to a second travelling-wave mode on the string, with qualitatively similar behaviour to, but different parameters from, the transverse motion. The torsional wave speed is generally higher than that of transverse waves, and there will be a pair of torsional reflection functions to describe the boundary reflections. It is also possible that boundary reflection produces some coupling between torsional and transverse motion, so that to characterise the reflection behaviour of, say, the bridge we would need a 2×2 matrix of reflection coefficients, and corresponding reflection functions. It is unlikely that significant coupling between the two wave types occurs during propagation, unless the string is damaged so that its cylindrical symmetry is broken.

3. Narrow reflection functions

The use of reflection functions to describe the transverse response of the string will prove very fruitful in the later discussions of minimum bow force and transient behaviour [1]. An interesting special case of the reflection-function description occurs if both reflection functions have significantly non-zero values only during a time interval which is short compared with the period of the note in question. Although this case is not very realistic, it is worth examining for several reasons. First, it allows some progress to be made in simple analytical calculations. Second, it is the case to which almost all existing published simulation results apply, so that understanding the limitations of the special case sheds light on those results. Third, Cremer has described a particular model coming within this class which fits at least some of the measured behaviour of a real instrument quite well [9, § 5.4]. Cremer's model makes a valuable benchmark against which the changes produced by more realistic models may be judged.

An obvious question left open by the string modelling described in the previous section is that of how the frequency of vibration should be determined from the driving-point response or the pair of reflection functions. The free modal frequencies and damping factors of the string will of course be given by the poles of $G(\omega)$, and these can be related to the reflection functions via eq. (3). However, for the case of narrow reflection functions an answer with more physical appeal may be found by considering the behaviour of a sinusoidal wave during one round trip on the string. Phase closure can be used to discover mode frequencies, and the amplitude decay rate gives the Q factors.

During a round trip, the sinusoidal wave is convolved once with each reflection function. We can consider these separately. The bridge reflection function, for example, will produce a certain phase shift and a certain amplitude reduction of the wave. To find the "effective reflection time" for this portion of the string at this frequency, we must find the time lag which produces the same phase shift. In other words, we want to find a position for the origin of time, presumably within the main peak of the reflection function, such that convolution produces *no* phase shift. For a function which is narrow compared with the period of the sinusoidal wave this is easily done, by considering the portion of the wave near a zero-crossing. Near $t=0$, for example, $\sin\omega t \approx \omega t$. For no phase shift we require that the zero crossing still occurs at $t=0$ after convolution, so that the effective delay t_1 must satisfy

$$\omega \int h_1(\tau)(t_1-\tau)\,\mathrm{d}\tau = 0 \tag{7}$$

where the integration is to be taken over the range where $h_1(\tau)$ is non-zero. The effective time delay is thus the one which makes the first moment of the reflection function vanish. Frequency does not influence this answer, so for all frequencies consistent with the approximation that $h_1(t)$ is narrow, the effective time delay is the same. The important result follows that *a narrow reflection function produces no anharmonicity.* Only frequencies high enough to have periods comparable with or shorter than the width of the reflection function can be influenced differentially.

This is of great importance for playability investigations. We may expect playability effects to arise from two aspects of string behaviour. First, energy dissipation is likely to play a role, as has been highlighted by Schelleng [11] and Cremer [9, § 4.6] in discussions of minimum bow force. Second, anharmonicity of string modes will presumably have an effect similar to that extensively investigated in wind instruments by Benade [12]. A model having narrow reflection functions contains the former effects without the latter (unless very high harmonics of the note are important). This makes such models interesting to study if we wish to understand the separate influences of these two effects.

Real instrument behaviour is certainly not well modelled by narrow reflection functions. As is made clear by eq. (6), the bridge reflection function will extend over a time governed by the Q-factors of the instrument body modes, and this is always several times longer than the period of even the lowest notes on the instrument. The nature of reflection from the player's finger is less easy to guess, but the nut/finger reflection function will in any case not be "narrow" if bending stiffness of the string is allowed for. It is well known that stiffness causes anharmonicity, and the above calculation shows that it therefore cannot be represented by a narrow reflection function. This is confirmed by analysis of wave propagation on a stiff string given in the Appendix, and is illustrated in Figs. (A1) and (A2).

A similar argument to that used to predict mode frequencies may be used to determine the damping factors of coupled string/body modes induced by narrow reflection functions. We can now ignore phase shifts, and calculate the reduction in amplitude of the peak of a sinusoidal wave due to convolution with each of the two reflection functions. It is convenient this time to consider the function $\cos\omega t \approx 1-\omega^2 t^2/2$

near the peak at $t=0$. The peak value after one convolution with $h_1(t)$ is found by evaluating the general convolution integral at the time t_1 satisfying eq. (7), with the approximate result

$$\int h_1(\tau)\left[1-\frac{\omega^2}{2}(t_1-\tau)^2\right]\mathrm{d}\tau=-(1-\varepsilon_1)+\omega^2\Delta_1^2 \quad (8)$$

where

$$1-\varepsilon_1=\int h_1(\tau)\,\mathrm{d}\tau,\ \Delta_1^2=-\frac{1}{2}\int(t_1-\tau)^2 h_1(\tau)\,\mathrm{d}\tau. \quad (9)$$

From the assumption that $h_1(t)$ is narrow, it follows directly that $\omega\Delta_1$ is a small quantity. The "width" of $h_1(t)$ is characterised by the value of Δ_1, which is independent of frequency. Quantities ε_2 and Δ_2 may be defined from $h_2(t)$ in exactly the same manner.

During one round trip on the string, the sinusoidal wave undergoes one convolution with each of $h_1(t)$ and $h_2(t)$. The net reduction in peak amplitude is then approximately $\varepsilon_1+\omega^2\Delta_1^2+\varepsilon_2+\omega^2\Delta_2^2$. For the nth harmonic of the note, the Q-factor thus satisfies

$$\frac{1}{Q_n}\approx\frac{[\varepsilon_1+4\pi^2 f_0^2 n^2\Delta_1^2+\varepsilon_2+4\pi^2 f_0^2 n^2\Delta_2^2]}{\pi n} \quad (10)$$

where the fundamental frequency of the note (in Hz) is given by

$$f_0=\frac{1}{t_1+t_2} \quad (11)$$

by the earlier result on modal frequencies.

Two special cases of models with narrow reflection functions are worth noting. First, if both reflection functions satisfy eq. (1) then ε_1 and ε_2 are both zero. This gives the behaviour

$$Q_n\sim 1/n \quad (12)$$

while n remains small enough for the assumption of narrow reflection functions to remain valid. Damping increases with mode number. A complete contrast is shown by the second case. The earliest dynamical model of the bowed string is due to Raman [13], and assumes an ideal string terminated in pure resistances at both ends. This produces reflection functions which are both delta functions, but with magnitudes which violate eq. (1). For this model, Δ_1 and Δ_2 are zero while ε_1 and ε_2 are not, so that eq. (10) yields

$$Q_n\sim n. \quad (13)$$

For Raman's model, damping decreases with mode number. The general case of narrow reflection functions combines the two types of damping behaviour: eq. (10) describes damping which decreases initially, then reaches a minimum at some value of n before increasing again.

The detailed predictions of Raman's model are known to be very unrealistic. The damping behaviour revealed by eq. (13) is one way of seeing why – this is the condition necessary for sharp corners in the velocity waveform to remain sharp for all time, which allows this waveform to evolve into strange "furry" forms which do not at all resemble the observed waveforms of real bowed strings. If instead at least one of the reflection functions has a finite width, a limit is set to this evolution, and more plausible waveforms can result. Computed examples of this contrast of waveforms are given in ref. [14]. (Raman was probably aware of this shortcoming of his model: he chose to catalogue possible periodic oscillation regimes according to the number of travelling corners on the string, and he stopped his detailed discussion at seven, presumably because he thought that other effects would prevent a large number of corners coexisting on a string [13].)

Several earlier papers on bowed-string dynamics were largely concerned with the implications of a change from Raman's model to a model with finite-width reflection functions – the so-called "rounded-corner" models [7–9, 15, 16]. This explains the emphasis in those papers on calculations and computer simulations using narrow reflection functions. Such models are the simplest which contain the main effect and achieve the great improvement in realism of predictions. Well-known examples of the success of these models are the explanations of the variation of velocity waveform with bow force [9, § 5.5] and of the "flattening effect" [7]. It was never imagined that such models represented the reflection functions of a real string on a real instrument with any great accuracy, but they made an ideal vehicle for exploring the effects of "corner-rounding" without added complications. In the days of smaller, slower computers, narrow reflection functions also allowed simulations to run with reasonable speed.

Finally, some remarks should be made about the contrasting behaviour shown by eqs. (12) and (13) and what happens at higher frequencies when the "narrow" approximation ceases to be valid. For any model having a reflection function which is smooth, the Q-factor will continue to fall with mode number even when the approximation leading to the linear behaviour of eq. (12) ceases to be valid. Eventually, for frequencies so high that the period is short compared with the time-scale of the reflection function, a single convolution will average the sinusoidal wave to a very small value, so that the decay rate becomes very rapid. This contrasts with the result for a model which contains a delta-function contribution as well as a narrow, smoother "tail" of some kind (for example, the reflection function resulting from Cremer's model, to be discussed next). In such a case the decay behaviour will eventually be governed by the delta function alone, and will follow the Raman-model behaviour of eq. (13) for high modes.

4. Cremer's model

The results of the previous section may conveniently be illustrated with a particular model based on narrow reflection functions, developed and used by Cremer in his well-known book [9]. He models the system as shown in Fig. 1, with an ideal string terminated in a perfect reflector at the nut/finger end and in a spring/dashpot combination representing the body. Cremer has given parameter values for this model, based on fitting it to experimental results on the time-decay behaviour of the overtones of an open A string on a violin. Since this data fit is quite good, Cremer's model can be regarded as about the best which can be achieved using narrow reflection functions. This makes it a very useful test case for computer simulations and other calculations, against which the changes produced by more realistic modelling may be judged. It is thus worth analysing in some detail here.

It is convenient to denote the value of the spring constant as μ/Y_0 and the dashpot rate as λ/Y_0, where Y_0 is the characteristic admittance of the string as before. Then the "body" admittance is

$$Y(\omega) = \frac{\mathrm{i}\,\omega\, Y_0}{\mu + \mathrm{i}\,\omega\,\lambda} \tag{14}$$

so that the reflection coefficient is

$$R_1(\omega) = \frac{\mathrm{i}\,\omega(1-\lambda) - \mu}{\mathrm{i}\,\omega(1+\lambda) + \mu}. \tag{15}$$

Note that for any model in which the string termination is fairly firm, either $\lambda \gg 1$ or $\mu/\omega \gg 1$ (or both), in which case $R_1 \approx -1$ as we would expect.

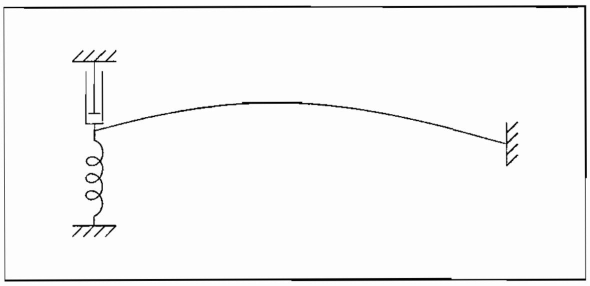

Fig. 1. Cremer's model of an ideal string terminated rigidly at one end, and in a spring-dashpot combination at the other end to represent the violin body.

The next step is to move to the time domain. For clarity, it is convenient to work in terms of the inverse Fourier transform of the reflection coefficient:

$$r_1(t) = \frac{1-\lambda}{1+\lambda}\,\delta(t) - \frac{2\,\mu}{(1+\lambda)^2}\,\mathrm{e}^{-\mu t/(1+\lambda)}\,(t \geq 0)\,. \tag{16}$$

This is related to the reflection function by a simple time delay:

$$h_1(t) = r_1(t - 2\,\beta\, L/c)\,. \tag{17}$$

The other reflection function is simply

$$h_2(t) = -\,\delta(t - 2(1-\beta)\,L/c)\,. \tag{18}$$

Both reflection functions satisfy eq. (1), as would be expected.

Several particular cases of eq. (16) are of interest. First, if the spring is removed so that $\mu = 0$ we have Raman's model in which the reflection function is just a delta function, as stated above. If, on the other hand, the resistance is removed leaving only the spring, we have

$$r_1(t) = \delta(t) - 2\,\mu\,\mathrm{e}^{-\mu t}\,(t \geq 0) \quad \text{(model with spring alone)} \tag{19}$$

which describes a purely reactive termination, with no energy dissipation. The initial delta function here is positive-going rather than negative-going, since for very high frequencies such a termination behaves like a free end rather than a fixed end to the string. Next, consider the case when $\lambda = 1$. Then, eq. (16) gives a reflection function with no initial delta function, only the decaying exponential. This is the "impedance-matched" case, where the resistive part of the termination matches the wave impedance of the string. It is not at all realistic for violin strings, but it is worth noting that Cremer used this very special case when illustrating an impulse response function in Fig. 8.5 of his book [9], presumably for clarity of drawing. Finally, note that if we assume $\lambda \gg 1$ we may obtain the approximate result

$$r_1(t) \approx -\,\delta(t) + \frac{2}{\lambda}\,\delta(t) - \frac{2\mu}{\lambda^2}\,\mathrm{e}^{-\mu t/\lambda} \tag{20}$$

which agrees with the general result of eq. (6).

The parameter values Cremer gives for this model, to fit decay data on a violin A string (440 Hz), are:

$$\lambda = 39; \quad \mu = 4 \times 10^5\,\mathrm{s}^{-1}. \tag{21}$$

If we wish to turn these into values for the spring constant and dashpot rate, we need a value for the characteristic admittance of the string. Measurements of string data by Pickering [17] give the following average values:

$$\begin{aligned}&\text{E string} - Y_0 = 5.7\ \mathrm{m\,N^{-1}\,s^{-1}};\\&\text{A string} - Y_0 = 5.5\ \mathrm{m\,N^{-1}\,s^{-1}};\\&\text{D string} - Y_0 = 4.4\ \mathrm{m\,N^{-1}\,s^{-1}};\\&\text{G string} - Y_0 = 3.1\ \mathrm{m\,N^{-1}\,s^{-1}}.\end{aligned} \tag{22}$$

Thus from the values (21) and this A-string admittance, Cremer's spring constant is about 7.3×10^4 N m^{-1}, and his dashpot rate about 7.1 N s m^{-1}.

With these parameters and a frequency of 440 Hz, the reflection function satisfies the "narrow" approximation for the first few overtones. Carrying out the integrals in eqs. (7) and (9), we obtain

$$t_1 = 2\beta L/c + 2/\mu; \quad \Delta_1^2 = 2\lambda/\mu^2 \tag{23}$$

while Δ_2^2, ε_1 and ε_2 are all zero. The value for t_1 gives the "end correction" for the string termination, which depends only on the spring, not the dashpot. The value of Δ_1^2 allows the modal Q-factors to be found approximately from eq. (10):

$$Q_n \approx \frac{\mu^2}{8\pi f_0^2 n \lambda}. \tag{24}$$

This may be compared with the more exact result obtained by carrying out the convolution integral on the full sinusoidal wave, without assuming narrowness:

$$Q_n = \frac{\mu^2 + 4\pi^2 n^2 f_0^2 (1+\lambda)^2}{8\pi f_0^2 n \lambda}. \tag{25}$$

This latter expression corresponds to the dashed line plotted by Cremer in his Fig. 2.5 [9]. As a result of the delta-function element of the reflection function, the Q-factors eventually grow with n, as explained in the previous section. With Cremer's numerical values, the assumption of narrowness is in fact only valid for the first few overtones and the minimum Q occurs for $n=4$.

5. Completing the model

We now examine briefly the remaining stages in constructing mathematical models of the bowing process. Most obviously, it is necessary to characterise the tribological behaviour of rosin so that the friction force may be related to the motion of the string. All published work on the bowed string up to now has used a very simple model for friction, but recent work has cast considerable doubt on the detailed validity of this model. We will call this model the "friction-curve model", since it assumes that frictional force depends only on the instantaneous value of relative velocity between bow and string according to a nonlinear relation of the kind plotted in Fig. 2. The vertical portion of the curve indicates that when bow and string are sticking, the force can take any value up to the limit of sticking friction. When there is relative sliding, the force is assumed to be uniquely determined by the sliding speed.

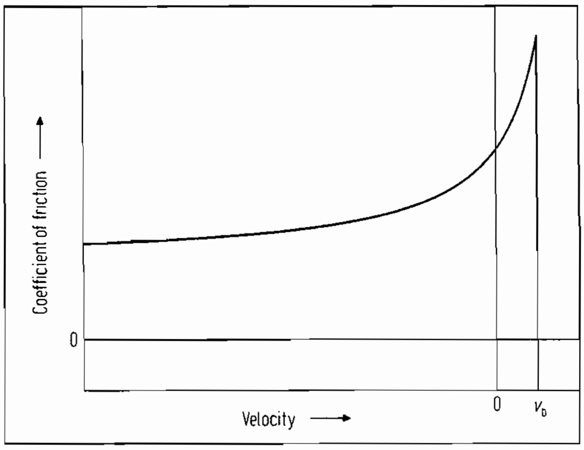

Fig. 2. Sketch of the form of dependence of frictional force on string velocity at the contact point usually assumed in the friction-curve model.

Although this model for friction has been so widely used in the past that it has come to be taken for granted, there is in fact no good physical reason why instantaneous sliding speed should be the only state variable governing the friction force. Of course, if measurements of the coefficient of friction are made in a steady-sliding apparatus, then the result can only depend on this parameter. However, in dynamic stick-slip oscillations other aspects of the motion and its history are likely to play a role. Smith [18] has recently studied the behaviour of a simple stick-slip oscillator to infer the actual variation of friction force with time, and he did not find a unique dependence on sliding speed. His apparatus was not operating in quite the same regime of contact conditions as prevails for the bowed string, so that it is hard to draw immediate quantitative conclusions for the present problem. However, the qualitative message is worth summarising.

Smith argues that the observed behaviour can be accounted for in terms of the thermal behaviour near the contact region. Rosin undergoes a glass transition not far above normal room temperature. Its viscosity then changes rapidly with temperature, so that a small amount of frictional heating at a sliding contact can reduce the frictional force, while a period of "sticking" may allow the contact to cool by diffusion, increasing the force again. The details and ramifications of this thermal model will be described in future papers. (Some direct observations of dynamic thermal variation under a bow have been made by Pickering [19].) For the moment we will use the friction-curve model, since it has considerable advantages of simplicity and computational efficiency, and it is known to give reasonably realistic predictions. However, we will maintain an awareness of its limitations. There is one aspect of Smith's results which can be used immediately: when actual values are needed for the coefficients of friction, the numbers found in static and steady-sliding tests may not be appropriate to the dynamic problem. At frequencies typical of the bowed string the contact will not have time to cool sufficiently during sticking to achieve the high values of sticking friction coefficient reported (up to 1.4 or so [20]), and a value of no more than 0.8 seems a reasonable guess for simulations based on a friction-curve model. Similarly, during the very brief slip time of a Helmholtz motion the contact may not heat up sufficiently to produce a coefficient as low as those found in steady-sliding tests.

One useful feature of the friction-curve model is worth recalling. So far we have neglected the influence of torsional string motion, except for some general remarks about how it can be included in a model. For preliminary studies of playability it may be acceptable to use a very crude representation of torsional behaviour. If it is assumed that torsional waves are so highly damped that reflected waves can simply be ignored, then torsion can be allowed for by a simple transformation of the effective friction curve, as has been described before [9, § 6.2]. This is obviously quite unrealistic, although Q-factors of torsional string modes are certainly much lower than those of transverse modes [10]. However a model using such a transformed curve does at least incorporate the essential energy-loss mechanism at the bowed point, due to scattering into torsional waves. This energy loss has a significant effect on certain aspects of the transverse motion, especially during transients. It tends to have a stabilising effect on the usual Helmholtz motion (although Weinreich and Caussé have recently pointed out that it does not influence all possible string motions relevant to a stability calculation [21]). It seems clear that to incorporate torsion in a crude way will be a great improvement on not allowing for it at all.

When all the approximations which have now been described are put together, the result is the simplest model which might perhaps display playability behaviour which is at least qualitatively similar to that of real instruments. The way in which this model can be used in computationally-efficient simulations has been described elsewhere [8]. The important approximations may be summarised as follows: neglect of the finite width of bow in contact with the string; neglect of motion of the string (and bow hair) in the plane orthogonal to the bowing plane; use of the simplistic friction-curve model for rosin tribology; and at least initially, use of a very crude method to represent the main effect of torsional string motion.

In the companion paper [1] this model is applied to the study of some specific issues related to playability. While there is no doubt that such studies have a long way to go before all aspects of playability of real instruments have been covered, the initial results are very encouraging and they suggest a number of promising avenues to follow in the immediate future.

Acknowledgements

The calculation described in the Appendix was carried out by Julian F. M. Scott. The author expresses profound gratitude to him, and to many other colleagues whose comments and insights have been invaluable. Chief among these are Robert T. Schumacher and Michael E. McIntyre.

Appendix

An approximate form is derived for the space and time response of a stiff string to a point force impulse. This may be used as an ingredient in the construction of more realistic reflection functions. The transverse displacement $y(x,t)$ of the string will be assumed to satisfy the usual approximate equation.

$$B\frac{\partial^4 y}{\partial x^4} - T\frac{\partial^2 y}{\partial x^2} + m\frac{\partial^2 y}{\partial t^2} = \delta(t)\,\delta(x)$$

where B, T and m are the bending stiffness, tension and mass per unit length respectively. Introduce non-dimensional variables

$$\bar{x} = \left(\frac{T}{B}\right)^{1/2} x, \quad \bar{t} = \frac{T}{(Bm)^{1/2}}\, t, \quad \bar{y} = (Tm)^{1/2}\, y$$

so that

$$\frac{\partial^4 \bar{y}}{\partial \bar{x}^4} - \frac{\partial^2 \bar{y}}{\partial \bar{x}^2} + \frac{\partial^2 \bar{y}}{\partial \bar{t}^2} = \delta(\bar{t})\,\delta(\bar{x}) \tag{A1}$$

with $\bar{y}=0$ for $\bar{t}<0$.

Taking Fourier transforms,

$$Y = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}k\bar{x}}\,\bar{y}\,\mathrm{d}\bar{x}, \quad \tilde{Y} = \int_{0}^{\infty} \mathrm{e}^{\mathrm{i}\omega\bar{t}}\, Y\,\mathrm{d}\bar{t}$$

where $\mathrm{Im}(\omega) > 0$ to guarantee convergence. Then from eq. (A1),

$$(k^4 + k^2 - \omega^2)\,\tilde{Y} = 1\,.$$

Inverting the transform with respect to ω,

$$Y=\frac{1}{2\pi}\int_{i\Omega-\infty}^{i\Omega+\infty}\frac{e^{-i\omega\bar{t}}}{k^4+k^2-\omega^2}\,d\omega.$$

For $\bar{t}>0$ this may be evaluated by residues to give

$$Y=\frac{\sin[k(k^2+1)^{1/2}\bar{t}]}{k(k^2+1)^{1/2}}.$$

Inverting the transform with respect to k gives the formal exact solution for $\bar{t}>0$:

$$\bar{y}=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\sin[k(k^2+1)^{1/2}\bar{t}]}{k(k^2+1)^{1/2}}\,e^{-ik\bar{x}}\,dk. \tag{A2}$$

For fixed values of x and t, as $B\to 0$, $\bar{x}$ and $\bar{t}\to\infty$. This is the limit which is now considered. It is most convenient to work with the velocity

$$\frac{\partial\bar{y}}{\partial\bar{t}}=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-ik\bar{x}}\cos[k(k^2+1)^{1/2}\bar{t}]\,dk. \tag{A3}$$

The integral may be evaluated approximately by the method of stationary phase. The exponential and cosine terms combine to give a sum of terms of the form $\exp(i\phi)$, where

$$\phi=-k\bar{x}\pm k(k^2+1)^{1/2}\bar{t}. \tag{A4}$$

Stationary phase points occur where $d\phi/dk=0$, at values of wavenumber k_p which satisfy

$$\frac{\bar{x}}{\bar{t}}=\pm\frac{2k_p^2+1}{(k_p^2+1)^{1/2}}=\pm c_g(k_p) \tag{A5}$$

where $c_g(k)$ is the non-dimensional group velocity, such that $c_g=1$ for the string in the absence of bending stiffness. Since we are assuming $\bar{x}>0$ and taking the positive square root, the minus sign in eq. (A4) does not produce a stationary phase point and can be ignored. There is also no stationary phase point for $\bar{t}>\bar{x}$, because $c_g\geq 1$ for all k. The regime where $\bar{t}$ is close to or greater than $\bar{x}$ will be investigated separately.

When there is a stationary phase point, eq. (A3) becomes

$$\frac{\partial\bar{y}}{\partial\bar{t}}\sim\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{i\phi}\,dk \tag{A6}$$

where

$$\phi=k(k^2+1)^{1/2}\bar{t}-k\bar{x}. \tag{A7}$$

Now expand in a Taylor's series around the stationary phase points $k=\pm k_p$:

$$\phi\sim\pm\bar{t}\left[\frac{k_p^3}{(k_p^2+1)^{1/2}}-\frac{k_p(2k_p^2+3)}{2(k_p^2+1)^{3/2}}(k\pm k_p)^2\right]$$

and eq. (A6) becomes

$$\frac{\partial\bar{y}}{\partial\bar{t}}\sim\mathrm{Re}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp\left[\frac{ik_p(2k_p^2+3)\bar{t}}{2(k_p^2+1)^{3/2}}(k-k_p)^2\right]dk\cdot\exp\left[\frac{-i\bar{t}k_p^3}{(k_p^2+1)^{1/2}}\right]\right\}$$

which may be evaluated as

$$\frac{\partial\bar{y}}{\partial\bar{t}}\sim\left[\frac{(k_p^2+1)^{3/2}}{2\pi k_p(2k_p^2+3)\bar{t}}\right]^{1/2}\cos\left[\frac{\pi}{4}-\frac{\bar{t}k_p^3}{(k_p^2+1)^{1/2}}\right] \tag{A8}$$

which is the final result from the stationary phase analysis. The solution of eq. (A5) for k_p may be written

$$k_p^2=\frac{1-4\tau^2+(1+8\tau^2)^{1/2}}{8\tau^2} \tag{A9}$$

where $\tau=\bar{t}/\bar{x}$.

Near $\bar{t}=\bar{x}$ the solution (A8) tends to infinity, and so obviously needs to be improved. The stationary phase points move towards $k=0$, and we expand ϕ about $k=0$. From eq. (A7) with $\bar{t}$ near to $\bar{x}$ and k small,

$$\phi\sim k(\bar{t}-\bar{x})+\frac{1}{2}k^3\bar{x}$$

and eq. (A6) yields

$$\frac{\partial\bar{y}}{\partial\bar{t}}\sim\frac{1}{4\pi}\int_{-\infty}^{\infty}e^{i[k(\bar{t}-\bar{x})+\frac{1}{2}k^3\bar{x}]}\,dk=\frac{1}{2}\left(\frac{2}{3\bar{x}}\right)^{1/3}\mathrm{Ai}\left[\left(\frac{2}{3\bar{x}}\right)^{1/3}(\bar{t}-\bar{x})\right] \tag{A10}$$

which describes the "front" in terms of the Airy function Ai. Matching this to the wavetrain solution (A8) we finally obtain a composite expansion

$$\frac{\partial\bar{y}}{\partial\bar{t}}\sim\frac{3^{1/6}(k_p^2+1)^{2/3}}{2^{2/3}(2k_p^2+3)^{1/2}\bar{t}^{1/3}}\cdot\mathrm{Ai}\left[-\left(\frac{3\bar{t}}{2}\right)^{2/3}\frac{k_p^2}{(k_p^2+1)^{1/3}}\right] \tag{A11}$$

which includes both cases analysed.

This expression may be computed easily. To show a representative example, it is first necessary to choose a value for the dimensionless distance $\bar{x}$. For a physical distance corresponding to one round trip on the string, it is easy to show that $\bar{x}$ depends only on the inharmonicity parameter called B by Schelleng [11], which for clarity we denote here B_s:

$$\bar{x}=\frac{\pi\sqrt{2}}{\sqrt{B_s}}. \tag{A12}$$

Schelleng suggests a maximum value of B_s for an acceptable string of 0.1/1731, which leads to a value $\bar{x}=584$. This is a large number, justifying the approximation used in deriving the results in this Appendix.

Since the length of the string does not enter eq. (A12), we may take the result based on this value to be representative of strings on a violin, cello or any other instrument, provided they satisfy Schelleng's suggested inharmonicity criterion.

The result of computing the velocity response from eq. (A11) with this value of $\bar{x}$ is shown in Fig. A1. As expected, a peak arrival occurs when $\bar{t} \approx \bar{x}$. Preceding that is the "precursor" due to the dispersion on the stiff string, with higher frequencies arriving progressively earlier and a singularity occurring at $t=0$ since the group velocity tends to infinity as frequency tends to infinity. Of course, the simple model of a stiff string used here is not to be believed for the extremely high frequencies which are responsible for the singularity at $t=0$, and the details of the result for very early times are of little physical significance. To obtain a picture more representative of what might be observed in practice, a suitable low-pass digital filter may be applied to the results of Fig. A1. A (somewhat arbitrary) choice of filter, described in the caption, yields Fig. A2, which shows behaviour quite reminiscent of what is seen on a real string.

An inverted version of Fig. A2 would make an interesting first candidate for a nut reflection function to represent string stiffness. It is not quite a reflection function as defined earlier, since it is calculated from the assumption of a point force impulse rather than from an initial delta function of velocity. The effects of stiffness around the driving point will make these two quantities different, but it is far from clear which is the more appropriate to use in the simple model of the bowing process described in this paper. Once such differences become important, the assumption of a point bow also becomes very unrealistic. One should consider the combined effects of string stiffness, finite bow width, and bow-hair and bow-stick compliance to represent the contact mechanics under the bow adequately. Such a study is under way, but lies beyond the scope of this paper.

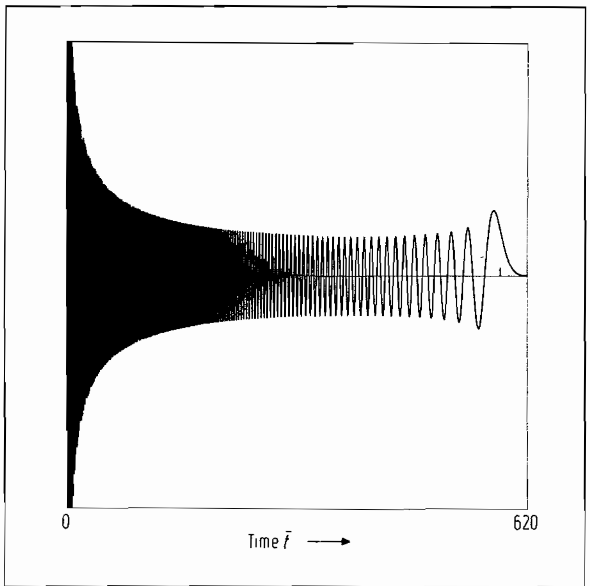

Fig. A1. Impulse response of a stiff string at a non-dimensional distance $\bar{x}=584$, calculated from eq. (A11). The time range is $0 \leq \bar{t} \leq 620$, and the tick on the time axis shows when $\bar{t}=\bar{x}$.

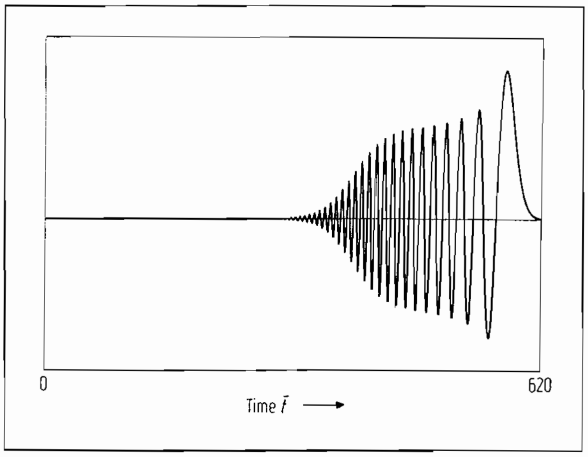

Fig. A2. Low-pass filtered version of Fig. A1, using a fifth-order Butterworth filter run forwards and then backwards through the data (to produce zero phase distortion). The assumed cutoff frequency is $75/T_W$ Hz, where T_W is the width of the time window plotted.

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