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# ON MEASURING WOOD PROPERTIES, PART 1

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## 1: Introduction

Violin makers are very fussy about wood. It is natural to try to quantify this fussiness by measuring the appropriate material properties of many wood samples, and trying to see what sets of values of those properties conform to traditional preferences in choosing wood. A related question is that of ageing: how do the material properties change with time, and with the various chemical treatments used by different people in the varnishing process? How might we measure all the relevant effects of wood selection and chemical treatments? The first stage in any such investigation is to establish how many quantities need to be measured, and how those measurements might be carried out.

Most past attempts on the problem have involved taking measurements on thin strips of wood, cut along and across the grain [1-4]. Measuring resonance frequencies of such strips enables one to deduce two elastic constants (the long-grain and cross-grain Young's moduli). If the damping of the strip vibration modes is also measured, with proper experimental caution, two corresponding damping constants can be deduced. This is by no means a complete set of measurements, however, since instruments are not generally built out of such strips. If instead of strips we consider, for example, flat, quarter-cut, thin plates of wood, then four elastic constants enter the problem, together with four corresponding damping constants [5,6].

Allowing for flat plates which are not accurately quarter-cut brings in yet more complication, and more elastic constants. Finally, to describe real violin plates, carved out of the solid with quite steep arching, as many as nine elastic constants enter the problem, with nine corresponding damping constants! Moreover, all these damping "constants" may well vary significantly with frequency of vibration within the audio range (as may the elastic constants in principle, although this seems less likely to present serious problems in practice) [7,8]. This now represents the full extent of the problem we need to consider, in the sense that it can be proved rigorously, from very general considerations of symmetry, that no more than this total of 18 material properties is needed to describe linear vibrations of an object of any shape carved from a given, homogeneous piece of wood, or other orthotropic material [5,9,10], at a given frequency.

The detailed theory of such linear vibrations is well understood and uncontroversial. At least in principle, it is known how to compute its consequences for any piece of wood of given shape, and any given values of the 18 material properties. Most of the efficient computational procedures, including those using the finite-element method [11], are based on a famous piece of mathematics known as "Rayleigh's variational principle". We believe that this principle is also the best basis for a general discussion of the problem, such as we attempt here, and it will be described in simple terms below.

In the main, existing work on the violin-wood problem has involved trying to guess a small subset of the complete set of material properties which might adequately characterise the desired qualities - e.g. two or three elastic and damping constants - and the development and empirical testing of heuristic or intuitively-based approximate formulae for particular geometries. (An interesting example of the latter approach, applied to the case of flat rectangular plates, appears in the article by Graham Caldersmith in this issue.) While such simplified theories have certainly been found useful in practice, it has never been made clear exactly what their limitations are: from a strictly mathematical standpoint, the approximations have a rather dubious and ad hoc character. One significant consequence is that it is far from clear what is needed to improve the accuracy of these simplified theories: one suspects that some suggestions for "improvements" which have been made over the years are inconsistent, in that some parts of the calculation may be improved while neglecting other parts giving rise to comparable errors. There is a need for a definitive study of the complete theory, together with accurate measurements of all the relevant material properties, to test the simplified theories against well-based computations which make no ad hoc approximations. It would then be possible to improve systematically on the approximations used in simplified theories, to the extent necessary (since of course one wants to have such simple theories available for practical purposes: the full theory is far too unwieldy for everyday use). Until such a definitive study is done, we shall not know for certain whether the baby has been thrown out with the bathwater in making any given simplification! It might turn out that violin quality is significantly affected by material parameters which have never yet been measured. After all, if one thing has become clear from the last few decades of violin research, it is that a great many small things contribute to violin quality, some of them unexpected and hard to quantify.

For reasons of length, this article will appear in instalments. In Part 1, we describe (with only a minimum of technical detail) the principles of the complete linear theory for mode frequencies and damping factors for the simplest interesting case, that of flat, thin, orthotropic plates under no static stress. This includes the cases of

(1) quarter-cut wooden plates;
(2) more general wooden plates cut at an angle to the grain or annual rings but with one symmetry axis of the wood still lying in the plate; and
(3) carbon-fibre composite sandwich plates.

We discuss a number of general aspects of the problem of measuring the properties of such plates, including the limitations of traditional strip measurements. Emphasis is laid on the advantages, both for intuitive understanding and for calculation, of studying the problem by the computationally efficient approach mentioned above, associated with the name of Rayleigh.

In Part 2, to follow, we shall describe some measurements made on free plates of soundboard spruce and carbon-fibre composite. These confirm aspects of the theory described in Part 1, and illustrate one approach to measuring the relevant material parameters of thin, flat plates. These measurements illustrate clearly the incompleteness of measurements using strips cut along and across the grain — undoubtedly valuable though such strip measurements are, particularly in view of their simplicity. The method and results described in Part 2 may also be of some practical value as they stand, particularly for suggesting approaches to the quality control of guitars and other instruments having flat soundboards.

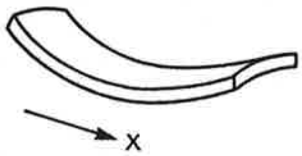
In Part 3, we shall discuss the wood measurement problem in its most general form. This is needed not only for conventional violin plates (carved out of the solid), but also for a proper understanding of flat plates which are not cut on the quarter. The experimental difficulties are formidable, if only because, as mentioned above, as many as 18 material properties need to be measured, separately at each frequency in principle, in order to understand that case in detail. Moreover, the properties will change significantly when static stresses are applied. We float some suggestions as to how measurements might be done, and point out some important technical pitfalls. So far as we are aware, no-one has yet even attempted a complete set of measurements on wood of musical instrument quality, although some measurements of all nine elastic constants on other kinds of wood have been reported: e.g. [5,12].

It is to be hoped that an experimenter of sufficient ingenuity will take up the challenge of carrying out such a complete measurement programme on wood specimens of good instrument quality, with and without various currently favoured varnishes and other chemical treatments. This would be a most valuable extension of existing strip-measurement programmes. Indeed, it could be argued that the problem of devising and carrying out suitable experiments is the single most important outstanding problem in violin acoustics today. It is certainly one of the most fundamental. Although the technical difficulties are formidable, as we have said and as we shall show in detail in Part 3, they need to be tackled before the problems of wood measurement, quality control, ageing and chemical treatments can be put on a fully scientific basis. As a step on the way, we hope to generate discussion as to what might be feasible with today's, or tomorrow's, technology.

## 2: Some basic theory: Rayleigh's principle

We now examine the problem of a flat, thin, orthotropic plate, and compare it with the more familiar case of a strip or beam. As everyone knows, the case of the strip involves just one elastic constant (Young's modulus E), together with the associated damping constant. The orthotropic plate can be shown (by very general symmetry arguments) to involve four independent elastic constants (plus four corresponding damping constants) [5,9]. The simplest way to see this is to consider the expressions for the elastic energy V involved in bending a small part of the beam, or the orthotropic plate, into a specified shape. For the beam, per unit length, this is simply

$$V = CE(w_{xx})^2 \qquad (1)$$

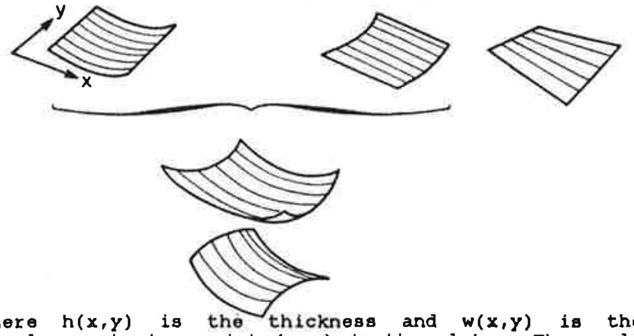



where C is a constant depending on the cross-sectional shape of the beam (for the case of a strip of small thickness h, it is $h^3/24$ times the strip width), E is Young's modulus, and w, a function of x along the beam, is the displacement of the centreline of the beam away from a straight, unstressed state of equilibrium. The notation $w_{xx}$ means the second derivative of the function w(x), a measure of the longitudinal curvature of bending. (It is equal to the reciprocal of the radius of curvature.) The small perspective drawing beneath the elastic energy expression is meant to suggest what the strip looks like when the curvature $w_{xx}$ is positive (w being positive for upward displacement). The amplitude is exaggerated for clarity. The simple fact that E is the only elastic constant appearing in the energy expression tells us immediately, without needing to do anything mathematical, that E is the only constant which can be measured via such bending vibrations of the strip.

It is worth noting also that the same simple expression (1) continues to hold even for a strip of variable thickness or width, or a beam of variable cross-sectional shape, so that C is a function of x. Such cases are sometimes regarded as far more complicated, and indeed they appear so when formulated in terms of a differential equation and boundary conditions.

The corresponding energy expression for a flat orthotropic plate contains more terms, because even within the assumptions of the standard thin-plate, bending-vibration theory the plate can be bent in more than one way. There are four terms, and four elastic constants which we shall denote $D_1$, $D_2$, $D_3$, $D_4$. Per unit area of the plate, the elastic energy is

$$V = \tfrac{1}{2}h^3\{D_1(w_{xx})^2 + D_2 w_{xx}w_{yy} + D_3(w_{yy})^2 + D_4(w_{xy})^2\} \qquad (2)$$




where h(x,y) is the thickness and w(x,y) is the displacement at a point (x,y) in the plate. The small perspective sketches beneath the expression illustrate the significance of the various terms, and will be discussed below.

The forms of the terms attached to the constants $D_1$ to $D_4$ in (2) indicate the physical significance of these constants. In the first term, the second derivative $w_{xx}$ represents bending in the x-direction, so $D_1$ is associated with the extent to which such bending contributes to the elastic energy of any general motion. Its effect would be felt in isolation from that of the other D's by a hypothetical motion like that sketched below the first term in (2), which is a pure bend in the x-direction. For such a motion, all the other terms in eq. (2) vanish. Similarly, y-direction bending is associated with $w_{yy}$ alone, and thus with $D_3$ as sketched. $D_4$ is associated with the pure twisting motion sketched ($w_{xx}=w_{yy}=0$, $w_{xy}\neq 0$), which, if it occurred, would isolate the effect of $D_4$ from that of the other D's. It happens that the lowest mode of vibration of a free rectangular plate is dominated by such twisting motion to a good approximation, so the sketch resembles such a mode fairly closely. (In consequence, the frequency and damping factor of this particular mode are determined almost entirely by the real and imaginary parts of $D_4$, as we shall explain shortly and illustrate from actual measurements in Part 2.)

There is no conceivable bending motion which can feel the effect of $D_2$ in isolation, since the $D_2$ term is non-zero only when bending occurs in both x and y directions simultaneously. Thus the $D_2$ term can contribute only when $D_1$ and $D_3$ are also contributing. This is not to say that it is unimportant, however. For example, if a ring mode exists for the plate in question, all three constants $D_1$, $D_2$ and $D_3$ will contribute significantly to determining the frequency and damping, as suggested by the ring-mode-like bending configuration sketched just below the horizontal brace, for which $w_{xy}=0$ but $w_{xx}$ and $w_{yy}$ are both non-zero, with the same sign. The same is true of the X-mode-like bending configuration (bottom sketch), but here the $D_2$ term contributes with a negative sign whereas for the ring mode its contribution was positive. (This is the essential reason for the lower frequency of the actual X-mode, as compared to the actual ring mode, as is made clear by Rayleigh's principle, which we now discuss).

The expression (2) is not only a nice way to picture what is going on in a bending plate, but it is also the simplest and most fundamental starting point for a rigorous theoretical investigation. It leads to methods which for many problems have better mathematical and computational properties than the direct use of the differential equation describing the plate vibration. The advantages become especially apparent when we go to complicated geometries [6]. No further complications ensue if, for example, the thickness h varies over the plate; the expression (2) already allows for this. It even allows for spatially-variable D's. (The differential equation and its boundary conditions, on the other hand, while giving a useful alternative viewpoint for some purposes, are quite complicated even for the simplest case of a rectangular plate of constant thickness, and become much more so for the case of a variable-thickness plate. They are given in the Appendix for reference purposes. Note, for example, the boundary condition (A6), which is an essential part of the "simplest" problem just mentioned!)

The integral of (2) over the area of the plate gives the total elastic (or potential) energy of the bending plate. Let us now introduce the kinetic energy, also an integral over the area of the plate, which is given by

$$\omega^2 T = \frac{1}{2}\omega^2 \iint \rho h w^2 dA \qquad (3)$$

where $\omega$ is (angular) frequency, $\rho$ is the plate density, and $dA = dx\, dy$, the area elements to be summed over. We now can use the principle mentioned earlier, which is usually associated with the name of Lord Rayleigh because he exploited it so effectively in "The Theory of Sound" [13] as well as proving it mathematically. This says firstly that if we can make a reasonably good guess at the mode shape, i.e. the function $w(x,y)$, for a particular vibration mode of our plate, then the ratio of the potential energy to the quantity T in Eq. (3), evaluated with that guessed mode shape $w(x,y)$, gives an estimate of the squared frequency of the mode:

$$\omega^2 = \frac{\iint h^3(D_1 w_{xx}{}^2 + D_2 w_{xx} w_{yy} + D_3 w_{yy}{}^2 + D_4 w_{xy}{}^2)dA}{\iint \rho h w^2 dA} \qquad (4)$$

Note how the elastic energy appears in the numerator. This makes precise the intuition that the stiffer the plate to the bending configuration of a particular mode, for a given mass $\rho h$ per unit area, the higher the frequency of vibration. Secondly (and this is the really important point), the expression (4) gives a far better estimate of $\omega^2$ than will the same guess for $w(x,y)$ substituted into the differential equation. The mathematical background is discussed in illuminating detail by Rayleigh [13].

The expression on the right-hand side of eq. (4) is called the "Rayleigh quotient", and the result that its value is insensitive to small errors in the mode shape $w(x,y)$ is known as "Rayleigh's variational principle" (or just "Rayleigh's principle"). This fact makes (4) the most efficient means of computing frequency (and damping), and also of studying their dependence on plate thickness, boundary conditions and material properties [6]. Moreover, it leads to well-known methods (e.g. the "Rayleigh-Ritz method") for systematically refining the approximation to $w(x,y)$ and thus obtaining estimates as accurate as desired.

The basic trick of getting good estimates of frequencies from less good estimates of mode shapes works only with (4), and not with the differential equation. We illustrate with a simple but instructive example. Consider the lowest mode of a free, rectangular plate of constant thickness and spatially uniform material properties. As we have already noted, the plate bends in approximately the manner sketched beneath the $D_4$ term in (2). The sketch represents the function $w(x,y) = xy$ (as indeed Rayleigh used as a guess for $w(x,y)$ in his book [13,§228]). Differentiating this function, we have $w_x = y$, $w_y = x$, $w_{xx} = w_{yy} = 0$, $w_{xy} = 1$. So only the last term in the numerator of (4) is non-zero: only $D_4$ is involved, to this approximation, as already noted. The denominator requires us to evaluate

$$\iint w^2\, dx\, dy = \iint x^2 y^2\, dx\, dy = \int_{-a/2}^{a/2} x^2 dx \int_{-b/2}^{b/2} y^2 dy$$

$$= \frac{1}{12}a^3\ \frac{1}{12}b^3,$$

where the dimensions of the plate are $a \times b$. Substituting into eq. (4) and taking the square root then yields an angular frequency for that mode

$$\omega = \frac{12h\sqrt{D_4}}{ab\sqrt{\rho}}$$

The frequency in Hz is $\omega/2\pi$. As discussed by Rayleigh [loc. cit.] in the isotropic case, this proves to be a reasonable estimate for the frequency. We shall illustrate it numerically for the orthotropic case in Part 2. If on the other hand we try the same guess $w = xy$ in the differential equation (A1) or (A5), it predicts a frequency of <u>zero</u>! Of course the differential equation <u>can</u> be used as a basis for finding $\omega$, but only if a far more accurate estimate of $w(x,y)$ is obtained.

The expression (4) confirms our earlier statement about how many material parameters we need to measure in order to characterise completely such a spatially homogeneous, orthotropic flat plate (within the scope of thin-plate theory). In the absence of dissipation, apart from the density (which is trivial to measure), we require the four elastic constants (which may perhaps vary significantly with frequency). To allow for damping, we need in addition to measure imaginary parts of the four elastic constants, which we may call damping constants. These latter certainly do vary with frequency [7,8], so measurements ideally should encompass the whole audio frequency range. (Calling the quantities elastic and damping "constants" is therefore something of a misnomer, but with some regret we follow what seems to be a well-established practice.)

If our wooden plate is accurately quarter-cut, it is straightforward to express the four constants $D_1$ to $D_4$ in terms of familiar quantities such as Young's moduli, Poisson's ratios, shear moduli and so on [5]. For the record, these expressions are given in the Appendix. However, it should be noted that while these other constants may have conceptual advantages in some problems, e.g. strip vibrations, they are by no means the most "natural" here. Any approach to the study of plate vibrations, be it analytic or finite-element numerical, starts from eq. (2), explicitly or implicitly, and therefore the most "natural" set of constants to use are arguably those which make that expression simplest. This choice of constants also makes the differential equation and boundary conditions least complicated - for instance, Hearmon (ref. [5], eq. (7.3.5)) makes essentially this choice. In any case, if our plate is not accurately quarter-cut, it is a far more complicated matter to express the D's in terms of the Young's moduli etc. of the solid material. This is an important issue in practice, as the data of Haines [3] shows, and it requires measurement of <u>all nine</u> elastic constants of the solid (and nine associated damping constants) before the D's can be deduced from any such formulae. We shall touch on the problem of the dependence of the plate parameters $D_1$ to $D_4$ on angle of tilt of the annual rings in Part 3.

As described above, when $w(x,y)$ is a vibration mode of the plate (with whatever boundary conditions are appropriate), the Rayleigh quotient (4) gives the squared angular frequency $\omega^2$ of that mode. Once we know to reasonable accuracy a particular mode shape and its frequency $\omega$, it is quite easy to extend the discussion to calculate the damping of that mode. It is convenient, and probably at least as accurate as any practical measurement technique, to use a "small damping" approximation which assumes that the modal Q-factor is much greater than unity, and to assume again that the D's are spatially homogeneous. We first introduce the four quantities $\eta_1$ to $\eta_4$, being the ratios of imaginary part to real part of $D_1$ to $D_4$ respectively (in other words the conventional loss factors associated with the D's). The reciprocal of the modal Q-factor of any given mode can be shown [6] to be simply a weighted sum of the four $\eta$'s. The formal expression is

$$Q^{-1} = \eta_1 J_1 + \eta_2 J_2 + \eta_3 J_3 + \eta_4 J_4\ , \qquad (5)$$

where

$$J_1 = \frac{D_1\iint h^3 w_{xx}{}^2 dA}{\omega^2\iint \rho h w^2 dA}, \qquad J_2 = \frac{D_2\iint h^3 w_{xx} w_{yy} dA}{\omega^2\iint \rho h w^2 dA},$$
$$J_3 = \frac{D_3\iint h^3 w_{yy}{}^2 dA}{\omega^2\iint \rho h w^2 dA}, \qquad J_4 = \frac{D_4\iint h^3 w_{xy}{}^2 dA}{\omega^2\iint \rho h w^2 dA}, \qquad (6)$$

so that (as can be seen immediately from (4))

$$J_1 + J_2 + J_3 + J_4 = 1\ . \qquad (7)$$

The justification of eqs. (5) and (6) again depends on Rayleigh's principle, and the proof is a simple mathematical exercise [6] which we do not reproduce here. The expressions (6) for the J's simply indicate the partitioning of potential energy, and thus dissipation rate, among the types of motion associated with each of the D's in eq. (2). They may be calculated readily if the vibration mode shape $w(x,y)$ and its (undamped) angular frequency $\omega$ are known. Equations (2)-(7) form an accurate and reliable basis both for measurements of the material properties of a flat orthotropic plate, and for applications of the measured values to predict the plate's behaviour. For example, we have used the method to study the effects of plate thickness adjustments [6].

### 3: Measurements using strips

As discussed earlier, most existing measurements of wood properties have been made using thin strips as specimens (e.g. [1-4]), cut along and across the grain. Strip measurements are relatively easy to perform, so we naturally enquire which of our constants can be measured in that way. Omitting some mathematical technicalities (concerned with minimising the elastic energy when a thin strip deforms anticlastically [6], as depicted in the sketch below eq. (1)) it turns out that for long, narrow strips cut from our thin, flat, orthotropic plate along the principal axes of the material, we can measure the real and imaginary parts of

$$E_1 = 12[D_1 - D_2^2/4D_3] \quad \text{and} \quad E_2 = 12[D_3 - D_2^2/4D_1] \qquad (8)$$

for strips cut along the x and y axes respectively. As we have already noted, $E_1$ and $E_2$ here are just the usual Young's moduli in the two directions (in Hearmon's notation [5], as used in the Appendix), expressed in terms of the fundamental constants $D_1$ to $D_4$.

As we have seen, two more (complex) constants must be measured before a full discussion of the elastic and damping properties of flat, quarter-cut, wooden plates may be attempted. Perhaps strips cut at different angles could be used? For a narrow strip cut from the plate at an angle $\theta$ to one principal axis (the x-axis), the constant whose real and imaginary parts we can measure is [6]

$$E_\theta = 3D_4(4D_1D_3 - D_2^2)/\Delta(\theta) \qquad (9)$$

where

$$\Delta(\theta) = D_4(D_1\sin^4\theta + D_3\cos^4\theta) + (4D_1D_3 - D_2^2 - D_2D_4)\sin^2\theta\cos^2\theta.$$

We note immediately from this last expression that $\sin\theta$ and $\cos\theta$ enter only in the three combinations $\sin^4\theta$, $\cos^4\theta$ and $\sin^2\theta\cos^2\theta$. It follows that only <u>three</u> independent combinations of the complex constants $D_1$ - $D_4$ can be measured from strips cut from a thin plate. For example, we might measure $E_1$, $E_2$ and $E_\theta$ for $\theta = \pi/4$. We should not be too surprised that strips cannot be used to determine all four of the plate constants: we already know for isotropic plates that only Young's modulus, and not Poisson's ratio, can be measured using strips (see the discussion following expression (1) and, for example, the experimental results in ref. [6], which remind us that Poisson's ratio certainly affects isotropic-plate vibrations).

Thus to determine all four of the D's, it is an uncomfortable but <u>inescapable</u> fact that we need to go beyond strip measurements. One possible approach, which requires no very sophisticated technology, is to study resonant vibrations of whole plates, and deduce real and imaginary parts of the D's from measured frequencies and Q-factors. In Part 2 of this article, we shall describe some preliminary measurements made in that way.

### Appendix

The equation of motion of a thin, flat plate lying in the x-y plane, with a slowly-varying thickness h(x,y) (and slowly-varying $D_1(x,y)$ to $D_4(x,y)$), is

$$\rho h w_{tt} + (D_1h^3w_{xx})_{xx} + \tfrac{1}{2}\{(D_2h^3w_{xx})_{yy} + (D_2h^3w_{yy})_{xx}\} + (D_3h^3w_{yy})_{yy} + (D_4h^3w_{xy})_{xy} = 0. \qquad (A1)$$

Each suffix denotes a differentiation; note very carefully that if any of h, $D_1$, $D_2$, $D_3$ or $D_4$ is a function of x and y there will be a large number of terms involving the derivatives $h_x$, $h_{xx}$, $h_{xy}$ and so on. The expanded form of this equation is, indeed, extremely complicated! As an example of boundary conditions appropriate to this equation, for a straight, free boundary $y = y_0$ = constant, parallel to the x-axis, one requires [5]

$$D_2w_{xx} + 2D_3w_{yy} = 0 \qquad (A2)$$

and

$$(D_2h^3w_{xx})_y + 2(D_3h^3w_{yy})_y + 2(D_4h^3w_{xy})_x = 0\,, \qquad (A3)$$

both relations to be satisfied at $y = y_0$. (See ref. [13] sections 223 and 224 for some intriguing history of past errors in deriving free-plate boundary conditions.) The boundary conditions on a boundary $x = x_0$ = constant have the same form, with x and y and $D_1$ and $D_3$ interchanged; and a subsidiary condition

$$w_{xy} = 0 \qquad (A4)$$

must be imposed at right-angled corners where two such boundaries meet, to express the physical fact that no external torques are applied at the corner. The proof from (4) is omitted; it uses the standard techniques of the calculus of variations. For a more complicated shape of boundary curve, these conditions would be more complicated still, though the same standard techniques will always reliably reveal them from the basic expression (4).

For a plate of <u>constant thickness</u> (and constant $D_1$ to $D_4$), eq. (A1) reduces to

$$\rho w_{tt} + h^2[D_1w_{xxxx} + (D_2+D_4)w_{xxyy} + D_3w_{yyyy}] = 0\,. \qquad (A5)$$

The sample boundary condition (A2) remains appropriate, while (A3) reduces to

$$2D_3w_{yyy} + (D_2 + 2D_4)w_{xxy} = 0. \qquad (A6)$$

Notice that in this case, while the equation of motion (A5) contains only three independent elastic constants, the factor 2 multiplying $D_4$ in (A6) implies that the fourth constant still enters the problem through the boundary conditions, a point which is sometimes overlooked.

For a flat plate cut from an orthotropic solid along one of the symmetry planes, the four constants $D_1$ to $D_4$ which we have used can be related easily to Young's moduli along and across the grain $E_1$ and $E_2$ (eq. (8) above), the two Poisson's ratios between the x and y directions $\nu_{12}$ and $\nu_{21}$, and the in-plane shear modulus $G_{12}$, in the standard notation used in the excellent book by Hearmon [5], eqs. (7.3.2):

$$D_1 = E_1/12\mu, \quad D_2 = \nu_{12}E_2/6\mu = \nu_{21}E_1/6\mu\,,$$
$$D_3 = E_2/12\mu, \quad D_4 = G_{12}/3, \qquad (A7)$$

where

$$\mu = 1 - \nu_{12}\nu_{21}.$$

The notation $\nu_{12}$ refers to the Poisson contraction in the 2 direction given a stretch in the 1 direction, while $\nu_{21}$ refers to the opposite case, again following the standard notational convention. There is a simple reciprocal property relating the two Young's moduli and the two corresponding Poisson's ratios, so that only three of these are independent, a fact used in exhibiting the symmetrical property of $D_2$ above:

$$\nu_{21}/E_2 = \nu_{12}/E_1. \qquad (A8)$$

This follows from a general reciprocal theorem discussed by Rayleigh [13,§72]. The fourth independent constant is the shear modulus $G_{12}$, which refers to <u>in-plate</u> shear, i.e. to the kind of shear in which a small square element on the top or bottom surface of the plate is deformed into a parallelogram, which is, of course, exactly what occurs in a pure twisting motion such as $w = xy$.

For a plate cut at an angle to the grain, far more complicated expressions, involving all the nine constants of the solid, would replace the expressions (A7). Since these expressions are of little practical interest, we omit them here.

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## A NOTE ON PRACTICAL BRIDGE TUNING FOR THE VIOLIN MAKER

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The proper fitting and trimming (or tuning) of a bridge for optimum sound is an extremely precise and critical part of violin making, requiring great skill as well as sensitivity to the sound qualities of the particular instrument at hand. I have found that some violin makers are very skillful at this so-called bridge tuning process, while others tend to cut a bridge to a given style and a set of measurements. In our shop where we are continually experimenting, it has been found best not only to have the maker able to play the violin, viola or cello fairly well, but also to have a good player at hand so that both can hear as well as feel the way the bridge tuning affects the sound and playing qualities in the particular instrument.

To understand a bit of how bridges bend let us review briefly some of the studies that have been done on bridge bending modes under different vibrating conditions.

In 1937 M. Minnaert and C. C. Vlam at the University of Utrecht used soft wax to fasten a galvanometer mirror of 0.025 grams onto a point of the bridge of a violin in normal playing condition. "A pencil of light, reflected by this mirror, was directed into a telescope, where the image of some very narrow pin-holes was formed. When a string was sounded, each luminous point described a curve, which did not alter in shape as long as the sound was constant." [1] They could bow the string and at the same time look through the telescope, or photograph the patterns traced by the luminous points. In this way they were able not only to show the motion in the plane of the bridge, but also flexural and torsional vibrations (Fig. 1).

Benjamin Bladier, in Marseille, studied the vibrations in several cello bridges, first on a block of concrete and then on the cello. He concluded that "the bridge is capable of governing and modifying the timbre of sound production in certain frequency ranges." [2]

Walter Reinicke, at the Heinrich Hertz Institute in Berlin, has done a theoretical study of the violin and cello bridge via transmission line theory using a circuit with two inputs and two outputs which is compared with actual holographic measurements.[3] When properly interpreted, the holograms of a violin bridge with rigidly supported feet show a rotational mode of the bridge top at about 3,000Hz (Fig. 2A). Those of a second mode at about 6,000Hz indicate that the horizontal members connecting the middle of the bridge to the feet act like springs, on which the upper part of the bridge bounces up and down (Fig. 2B).

Reinicke's holograms of a rigidly supported cello bridge show bending modes at 985Hz, 1450 Hz, and 2100Hz. At 985Hz the upper and middle portions sway on the supple legs. At 1450Hz there is a bending of the middle of the bridge with somewhat less bending of the legs. At 2100Hz there is a rotation of the upper portion about an axis near the heart similar to the first mode of a violin bridge (Fig. 3).

Helmut Müller, who teacher acoustics at the Geigenbauschule, Mittenwald, discusses the motions of the violin bridge as indicated by Reinicke and illustrates the effects of changes

Flexural and torsional bending

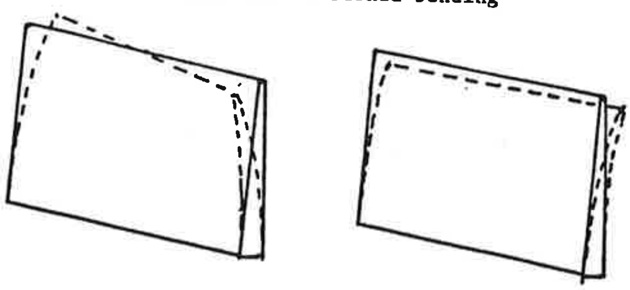

Figure 1
(after Minnaert and Vlam)